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Notes on long-crested water waves

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Fully three-dimensional surface gravity waves in deep water are investigated in the limit in which the length of the wave crests become long. We describe an analytic solution to fourth order in wave steepness, which matches onto known short-crested wave solutions on the one hand and onto the well-known two-dimensional progressive-wave solution on the other. In the progressive-wave limit a particular solution in which the wave crests are semi-finite is given to sixth-order accuracy. These solutions are part of a more general set of solutions which are found from a nonlinear Schrödinger equation.

1. Introduction

Short-crested water waves, investigated in the companion paper by Roberts (1983); hereinafter referred to as (I)), can be identified by two dimensionless parameters; the wave steepness h and an angle θ characterizing the aspect ratio of the basic periodic rectangle. In the limit as θ tends to 90° the waves become long-crested (see figure 7(*d*) in (I) for such a wave) and it is the form of the waves in this limit which is examined in this paper. The waves are not aptly described by the amplitude expansion discussed in (I) because the radius of convergence of the expansion tends to zero as $\theta \to 90°$. This is due to the divisors of the $(1, n)$ harmonic coefficients (that is the $\cos[1(px-\omega t)]\cos[nqz]$ terms) which all tend to zero (see tables 1 and 2 in (I)). These small divisors for θ near 90° can be attributed to the presence of strong singularities occurring at complex h; their location, given approximately by the empirical relation

$$h = \pm e^{\pm\frac{1}{4}i\pi}[1.82\beta+0.16\beta^2], \tag{1.1}$$

where

$$\beta = \tfrac{1}{2}\pi-\theta, \tag{1.2}$$

moves in to the origin as θ tends to 90°. These singularities, unlike the singularities occurring due to harmonic resonance in short-crested waves, are not simple poles. However, the singularities lie off the real h-axis and hence the Padé transform (for example) will produce converged answers for real h. Thus at finite height there is a continuous variation between short-crested waves and the longest long-crested wave, and *vice versa*.

The structure of long-crested waves is analysed by assuming that derivatives in the direction Oz along the crest are small compared with those in the direction Ox

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of propagation. A similar approach is used by Whitham (1976) to obtain a nonlinear approximation to edge waves propagating along a nearly vertical beach. To the same order, solutions described here involve only a slight variation to Whitham's equation (96). The solutions that are doubly periodic and, as in (I), correspond at first order to the superposition of two wavetrains of equal amplitude and frequency, have the form

$$\operatorname{sn}(hz/m^{\frac{1}{2}}|m)\cos(x-t), \tag{1.3}$$

where the notation of Abramowitz & Stegun (1965) is used for the Jacobian elliptic function. In the case of long period in z, i.e. $m\rightarrow 1$, waves are closely two-dimensional except near the ends of each crest. There the slopes in the z-direction are $O(h^2)$ instead of the $O(h^3)$ one would expect from the linear solution. At the other extreme, small m, the solution approaches that for short-crested waves.

To match onto the short-crested solution of (I) it is necessary to modify Whitham's (1976) equation (96) slightly; this is done in §2. In §3 the calculation is analytically extended to the fourth order in the wave steepness. Although it is only valid over part of the parameter regime this solution is currently the highest-order analytic formula available for calculating the properties of short-crested waves.

In the limit as wave crests become very long, two distinct solutions are possible. If both ends of the crest are at a very large distance, the limit is the two-dimensional progressive wave. On the other hand, if one end of the crests is kept at a finite position a semi-infinite-crested wave is obtained. Along a line perpendicular to the crests there is a phase jump of one half-period between two uniform wavetrains (see figures 4 and 5). This solution corresponds to the soliton of maximum amplitude for the 'defocusing' nonlinear Schrödinger equation (Zakharov & Shabat 1973). These solitons were first studied in optics, and are called 'dark' solitons since the wave amplitude is lower in the soliton than in the carrying wavefield.

In §5 these long-crested solutions are placed in the wider context of slow variations of a nearly uniform wavetrain. A substantial review of this area is given by Yuen & Lake (1982); we pick out only those aspects of particular relevance to long-crested waves and wave reflection.

2. Leading-order solution

Following (I) we non-dimensionalize the problem and then introduce the scaled independent variables

$$X = px-\omega t, \quad Z = qz, \tag{2.1}$$

where $p = \cos\beta$, $q = \sin\beta$, and ω is the frequency of the wave. We look for a surface shape $\eta(X,Z)$ and a velocity potential $\phi(X,y,Z)$, periodic in X and Z with period 2π, such that the velocity potential satisfies the transformed Laplace equation in the body of the fluid

$$p^2\phi_{XX}+\phi_{yy}+q^2\phi_{ZZ} = 0 \quad (y<\eta(X,Z)). \tag{2.2}$$

There are two boundary conditions at the free surface: the kinematic condition that no fluid crosses the surface can be written as

$$-\omega\eta_X+p^2\eta_X\phi_X-\phi_y+q^2\eta_Z\phi_z = 0 \quad (y = \eta(X,Z)), \tag{2.3}$$

while the condition of constant pressure is

$$-\omega\phi_X+y+\tfrac{1}{2}(p^2\phi_X^2+\phi_y^2+q^2\phi_Z^2) = 0 \quad (y = \eta(X,Z)). \tag{2.4}$$

Infinitely deep in the fluid there must be no motion, and so we enforce the necessary condition

$$\phi_y \to 0 \quad \text{as} \quad y \to -\infty. \tag{2.5}$$

To derive the long-crested wave solution, we suppose $\beta \ll 1$ and find that the appropriate scaling is $\beta = O(h)$. Hence the wavenumbers in the x- and z-directions may be written

$$q = \sin\beta = Qh, \quad p = \cos\beta = (1-Q^2h^2)^{\frac{1}{2}}, \tag{2.6}$$

where Q is now the aspect-ratio parameter. We also expand the unknown functions in powers of the wave steepness:

$$\eta = \sum_{r=1}^{\infty} h^r \eta_r(X,Z), \quad \phi = \sum_{r=1}^{\infty} h^r \phi_r(X,y,Z), \quad \omega = \sum_{r=0}^{\infty} h^r \omega_r. \tag{2.7}$$

Upon substituting (2.6) and the assumed perturbation form (2.7) for the dependent variables into the governing differential equation (2.2) and the boundary conditions (2.3)–(2.5), we obtain a set of equations which may be successively solved for the unknown functions. The equations are of the form

$$\left.\begin{aligned} \phi_{r_{XX}} + \phi_{r_{yy}} &= Q^2\phi_{(r-2)_{XX}} - Q^2\phi_{(r-2)_{ZZ}}, \\ \phi_{r_y} &\to 0 \quad \text{as} \quad y \to -\infty, \\ \omega_0 \eta_{r_X} + \phi_{r_y} &= F_r, \\ -\omega_0 \phi_{r_X} + \eta_r &= G_r \quad (r = 1,2,3,\ldots), \end{aligned}\right\} \tag{2.8}$$

where ϕ_{-1} and ϕ_0 are defined to be zero and where F_r and G_r are some nonlinear combinations of the lower-order functions (note that $F_1 = G_1 = 0$).

The first-order linear solution to (2.8) is simply

$$\left.\begin{aligned} \omega_0 &= 1, \\ \eta_1 &= H_1(Z)\cos(X), \\ \phi_1 &= H_1(Z)\sin(X)\exp(y), \end{aligned}\right\} \tag{2.9}$$

where the as-yet-arbitrary function H_1 will be determined by a secularity condition obtained at a higher order. In general, at the rth order the forcing F_r and G_r will contain some component in $\sin(X)\exp(y)$ which would force secular terms into the expression for ϕ_r and η_r. Setting the coefficient of this forcing to zero and using the definition of the perturbation parameter and the periodicity gives us a second-order differential equation and eigenvalue problem for H_{r-2} and ω_{r-1}. Also at the rth order we will have to introduce a homogeneous solution of the linear problem into the expressions for ϕ_r and η_r, $H_r(Z)\sin(X)\exp(y)$ and $H_r(Z)\cos(X)$ respectively.

At second order the non-secularity condition just gives $\omega_1 = 0$. However, at third order we obtain the nonlinear equation

$$Q^2H_1'' + (4\omega_2 + Q^2)H_1 - 2H_1^3 = 0, \tag{2.10}$$

where the primes denote derivatives with respect to Z. This equation has the same form as equation (96) of Whitham (1976). The differences are due to the specific periodicity in z and the variation of the x-direction wavelength implied by (2.6). Equation (2.10) may immediately be integrated once to give

$$Q^2H_1'^2 + (4\omega_2 + Q^2)H_1^2 - H_1^4 - D = 0, \tag{2.11}$$

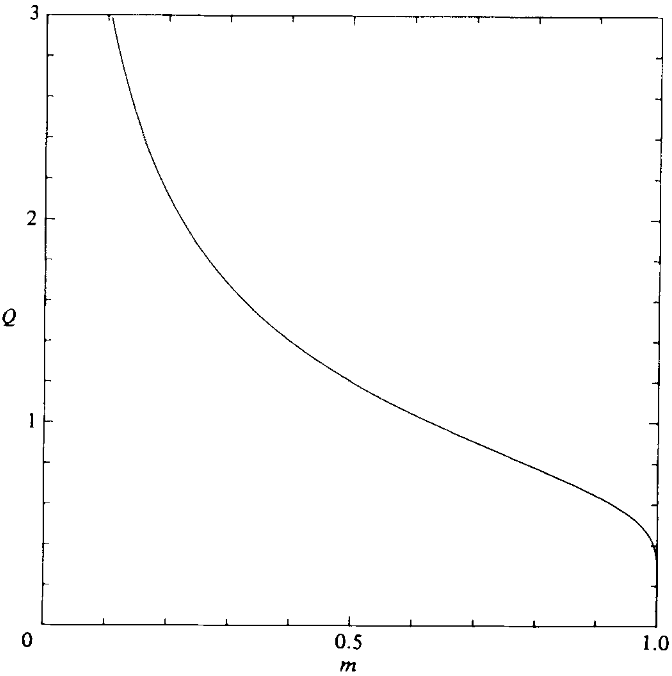

FIGURE 1. The relation between the long-crested wave parameter Q and the elliptic function parameter m, from (2.13).

where D is the constant of integration. The only real-valued solution of this equation which has the required periodicity is

$$\begin{aligned} H_1(Z) &= \operatorname{sn}(2KZ/\pi\,|\,m), \\ \omega_2 &= \tfrac{1}{4}[1+1/m-Q^2], \\ D &= 1/m, \end{aligned} \tag{2.12}$$

where

$$Q^2 = \pi^2/4mK^2, \tag{2.13}$$

and $K = K(m)$ is the complete elliptic integral of the first kind (see figure 1 for a graph of Q versus m). Equation (2.13) implicitly gives the elliptic function parameter m in terms of the parameter Q. For values of Q which are small it can be seen that m is very close to 1; asymptotically we find

$$m \to 1 - 16\exp(-\pi/Q) \quad \text{as} \quad Q \to 0.$$

In the limit as Q gets large, that is $m \to 0$, (2.13) gives

$$Q^2 = 1/m - \tfrac{1}{2} - \tfrac{3}{32}m + O(m^2),$$

and so

$$H_1 \to \sin(Z), \quad \omega_2 \to \tfrac{3}{8} \quad \text{as } m \to 0, \tag{2.14}$$

which agrees with the short-crested wave solution (apart from the trivial shift in the Z-dependence). Provided that the higher orders in this long-crested wave expansion can be calculated, we see that there is indeed a continuous limit from finite-amplitude short-crested waves to these long-crested waves (see figure 3 for a sequence of wave profiles illustrating this transition). The reason that zero-divisors occur as $\theta \to 90°$

in the short-crested wave amplitude expansion of (I) is that the perturbation expansion implicitly assumes that higher Fourier harmonics are only generated at higher orders in the expansion. But a Fourier decomposition of an elliptic function requires order-1 Fourier coefficients (although they may be numerically small). Thus as $\theta \rightarrow 90°$ the coefficients of the higher harmonics of the short-crested wave expansion become large in an endeavour to make their appropriate contribution (order h) to the solution. This reasoning is quite general and explains equally well the behaviour of the coefficients near other harmonic resonances.

From this solution we can calculate to leading-order accuracy the mean energy density in long-crested waves. It is

$$\mathrm{KE} = \mathrm{PE} = \tfrac{1}{4}h^2\left(1-\frac{E}{K}\right)\frac{1}{m}. \tag{2.15}$$

For small m this reduces to $\frac{1}{8}h^2$, which is the expression for linear short-crested waves; for m near 1 it reduces to $\frac{1}{4}h^2$, which is the appropriate expression for two-dimensional progressive waves. Equation (2.15) describes the initial upwards bend of the S-shaped energy curves for long-crested waves (see figure 6(e) in (I)).

3. Fourth-order analytic solution

Carrying the expansion to the fourth order we find an equation relating H_2 and ω_3 with the solution

$$H_2(Z) = 0, \quad \omega_3 = 0, \tag{3.1}$$

as expected. Similarly we expect that $H_{2r} = \omega_{2r+1} = 0$ for $r = 2, 3, 4, \ldots$. At the fifth order the non-secularity condition gives the following linear equation for H_3 and ω_4:

$$Q^2H_3'' + (4\omega_2 + Q^2 - 6H_1^2)\,H_3 = g_3(Z;\omega_4), \tag{3.2}$$

where

$$g_3(Z;\omega_4) = -(4\omega_4 + 6/m + 3\omega_2^2)\,H_1 + 16\omega_2 H_1^3 - 3H_1^5. \tag{3.3}$$

At higher orders in this expansion the equation that H_r has to satisfy to ensure non-secularity will be of the same form as (3.2) but with different forcing, $g_r(Z;\omega_{r+1})$ say, on the right-hand side. An equation such as (3.2) can be integrated twice by using (2.10). The general solution is

$$H_r = \gamma_r H_1' + \frac{H_1'}{Q^2}\int \frac{I_r(Z;\omega_{r+1},\beta_r)}{H_1'^2}\,\mathrm{d}Z, \tag{3.4}$$

where

$$I_r = \beta_r + \int g_r(Z;\omega_{r+1})\,H_1'\,\mathrm{d}Z, \tag{3.5}$$

with β_r and γ_r constants of integration. The value of β_r is chosen to eliminate the double pole singularity in the integrand of (3.4) at $Z = \frac{1}{2}\pi$. Since I_r will be a symmetric function about $Z = \frac{1}{2}\pi$, this choice of β_r will ensure that the integral in (3.4) is integrable. The value of ω_{r-1} is then chosen so that the function H_r satisfies the periodicity requirements. Finally γ_r is chosen so that $H_r(0) = 0$, as any other choice will lead to a trivially different solution where the phase in the Z-direction varies with amplitude. Hence the solution of (3.2) and (3.3) is

$$\left.\begin{aligned} H_3 &= \frac{K}{E}(Q^2-\tfrac{1}{2})\,\mathrm{cn}\,(u)\,\mathrm{dn}\,(u)\,Z(u) + \left[\tfrac{1}{2} - m\frac{K}{E}(Q^2-\tfrac{1}{2})\right]\mathrm{sn}\,(u)\,\mathrm{cn}\,(u)^2, \\ \omega_4 &= \frac{K}{E}\frac{(2Q^2-1)(1-m)}{4m} - \frac{1}{32}\left[3Q^4 + 10\left(1+\frac{1}{m}\right)Q^2 + \frac{3}{m^2} - \frac{2}{m} - 5\right], \end{aligned}\right\} \tag{3.6}$$

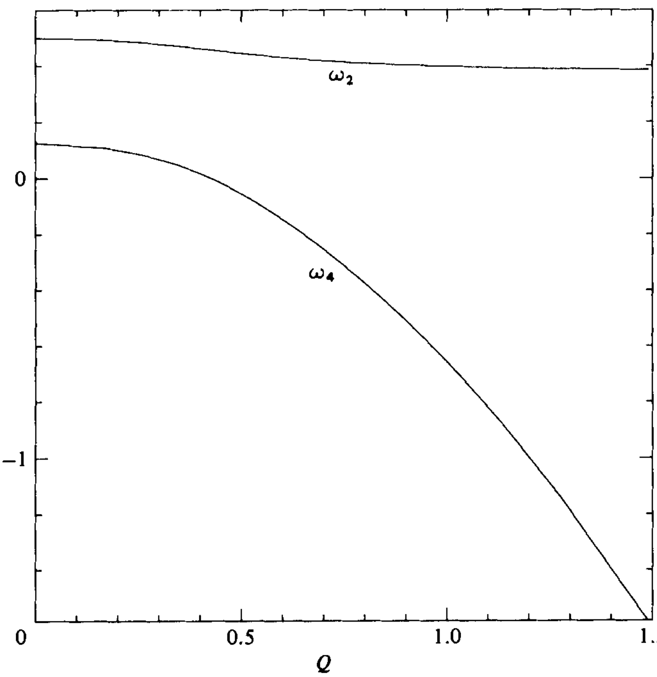

FIGURE 2. The first two coefficients of the frequency-correction expansion (2.7) as a function of the parameter Q.

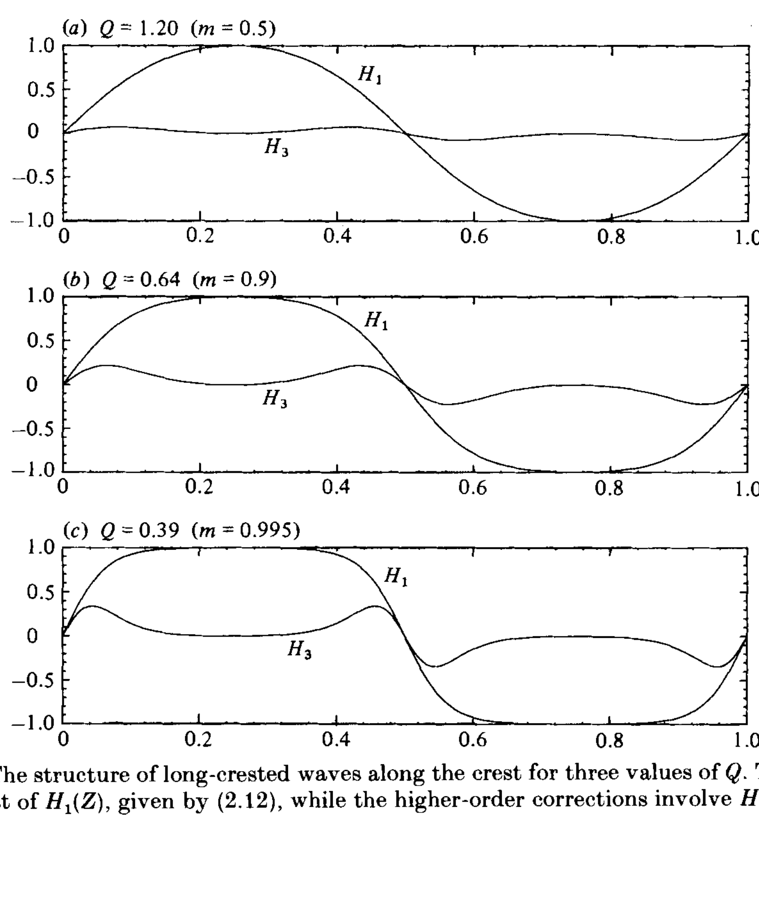

FIGURE 3. The structure of long-crested waves along the crest for three values of Q. The dominant shape is that of $H_1(Z)$, given by (2.12), while the higher-order corrections involve $H_3(Z)$, given by (3.6).

where $u = 2KZ/\pi$, $E = E(m)$ is the complete elliptic integral of the second kind and $Z(u)$ is Jacobi's zeta function, which will be distinguished from the coordinate Z by always including the argument of the function. The above analytic solution has been confirmed by calculating numerical solutions of (3.2). A plot of ω_2 and ω_4 versus Q is shown in figure 2. The appreciable flattening of the crest of the leading-order z-direction profile given by H_1 is enhanced, especially for small Q, by the correction due to H_3 (figure 3).

The fourth-order solution of the long-crested system of equations, in terms of the quantities defined by (2.12), (2.13) and (3.6), is

$$\begin{aligned}
\omega &= 1+\omega_2 h^2+\omega_4 h^4+O(h^6),\\
\phi &= \{hH_1+h^3[H_3-\omega_2 H_1-y(H_1^3-2\omega_2 H_1)]\}\sin(X)\exp(y)\\
&\quad +h^4(H_1^4-\tfrac{1}{2}D)\sin(2X)\exp(2y)+O(h^5),\\
\eta &= h^4[-\tfrac{3}{4}H_1^4+(2\omega_2+\tfrac{1}{2}Q^2)H_1^2-\tfrac{1}{4}D]\\
&\quad +\{hH_1+h^3[H_3-\tfrac{3}{8}H_1^3]\}\cos(X)\\
&\quad +\{h^2\tfrac{1}{2}H_1^2+h^4[H_1H_3+\tfrac{7}{12}H_1^4+\omega_2 H_1^2-\tfrac{3}{4}D]\}\cos(2X)\\
&\quad +h^3\tfrac{3}{8}H_1^3\cos(3X)+h^4\tfrac{1}{3}H_1^4\cos(4X)+O(h^5).
\end{aligned} \tag{3.7}$$

Incidentally this solution is currently the highest-order analytic solution available for calculating finite-crested waves, although infinitesimal short-crested waves are not represented as well as they are by other analytic perturbation expansions.

4. Semi-infinite-crested wave

In the limit as $Q \to 0$ we obtain a train of gravity waves propagating in the x-direction with a 180° phase shift occurring in the x-dependence in the vicinity of $z = 0$ (see figures 4 and 5). In the z-direction each wave crest starts in the vicinity of the phase jump and extends to infinity in one direction; hence the name 'semi-infinite-crested wave'. Alternatively, if the origin of z is taken at the centre of a wave crest then the limit $Q \to 0$ gives a two-dimensional wave.

Rather than take the limit of the long-crested wave solution (3.7), it is more convenient to derive the solution afresh from the equations. We assume the z-direction derivatives are of order h and so set $q^2 = h^2$ and $p^2 = 1$. Substituting this and the assumed perturbation expansion (2.7) into the governing equations and grouping like powers of h, we can then solve the resulting set of equations in succession. These equations were solved analytically to find the solution up to sixth-order quantities. Writing, for brevity, $T = \tanh(Z)$, we find

$$\omega = 1+\tfrac{1}{2}h^2+\tfrac{1}{8}h^4+\tfrac{1}{16}h^6+O(h^8), \tag{4.1}$$

$$\begin{aligned}
\eta &= hT\cos(x)\\
&\quad +h^2\tfrac{1}{2}T^2\cos(2x)\\
&\quad +h^3\{[-\tfrac{3}{8}T^3+\tfrac{1}{2}(1-T^2)(Z+T)]\cos(x)+\tfrac{3}{8}T^3\cos(3x)\}\\
&\quad +h^4\{\tfrac{1}{4}(1-T^2)(3T^2-1)+[\tfrac{1}{12}(T^4+12T^2-9)+\tfrac{1}{2}(1-T^2)ZT]\cos(2x)\\
&\quad\quad +\tfrac{1}{3}T^4\cos(4x)\}
\end{aligned}$$

$$
\begin{aligned}
&+h^5\{[\tfrac{1}{576}(-1217T^5+3740T^3-3156T)\\
&\quad+\tfrac{1}{48}(1-T^2)(172Z-63ZT^2-12Z^2T)]\cos(x)\\
&\quad+[\tfrac{1}{128}(75T^5+168T^3-144T)+(1-T^2)\tfrac{9}{16}ZT^2]\cos(3x)+\tfrac{125}{384}T^5\cos(5x)\}\\
&+h^6\{\tfrac{1}{4}(1-T^2)(-1+6T^2-5T^4+4ZT-6ZT^3)\\
&\quad+[\tfrac{1}{144}(-2T^6+716T^4-789T^2+36)\\
&\quad+\tfrac{1}{24}(1-T^2)(104ZT+4ZT^3+3Z^2-9Z^2T^2)]\cos(2x)\\
&\quad+[\tfrac{1}{180}(187T^6+300T^4-270T^2)+(1-T^2)\tfrac{2}{3}ZT^3]\cos(4x)+\tfrac{27}{80}T^6\cos(6x)\}\\
&+O(h^7),
\end{aligned}
\tag{4.2}
$$

$$
\begin{aligned}
\phi = {}& hT\sin(x)\exp(y)\\
&+h^3[-\tfrac{1}{2}T^3+(1-T^2)(yT+\tfrac{1}{2}Z)]\sin(x)\exp(y)\\
&+h^4[T^4-\tfrac{1}{2}]\sin(2x)\exp(y)\\
&+h^5\{[\tfrac{1}{144}(-269T^5+728T^3-681T)+(1-T^2)(\tfrac{1}{2}yT\\
&\quad+\tfrac{1}{12}Z(40-9T^2)+y^2(2-3T^2)\\
&\quad+\tfrac{1}{2}yZ(1-3T^2)-\tfrac{1}{4}Z^2T)]\sin(x)\exp(y)\\
&\quad+\tfrac{1}{12}T^5\sin(3x)\exp(3y)\}\\
&+h^6\{[\tfrac{1}{12}(8T^6-6T^4-3T^2+3)\\
&\quad+T^2(1-T^2)(-y(3-5T^2)+2ZT)]\sin(2x)\exp(2y)\\
&\quad+\tfrac{1}{72}T^6\sin(4x)\exp(4y)\}\\
&+O(h^7).
\end{aligned}
\tag{4.3}
$$

In the limit as $Z\to\infty$, where the solution looks like a uniform train of waves, we indeed find agreement with well-known analytical expansions for infinite-crested Stokes waves (see De 1955). In particular, the series for the frequency correction is identical with that for two-dimensional waves.

5. Discussion

The solutions described above are the natural extension of the short-crested waves discussed in (I). This whole class of waves may be interpreted as those wavefields obtained by the perfect reflection of a uniform wavetrain from a straight reflector, e.g. a vertical wall. The case considered in this paper corresponds to waves which are incident nearly parallel to the reflector, i.e. $\beta \ll 1$.

The physical realization of short-crested waves by reflection is straightforward. However, long-crested waves are more difficult to create in this way. Their wave pattern has a wavelength $2\pi/k\beta$ normal to the reflector; hence linear ray theory shows that at least a length $2\pi/k\beta^2$ of reflector is necessary before any regular wave pattern can be established. At that distance the linear diffractive effects associated with the edge of the reflected region also have an extent $O(1/k\beta)$, indicating that a somewhat greater distance is required. In fact Yue & Mei (1980) show that steady conditions near a reflecting wall are established more quickly for nonlinear waves than for linear

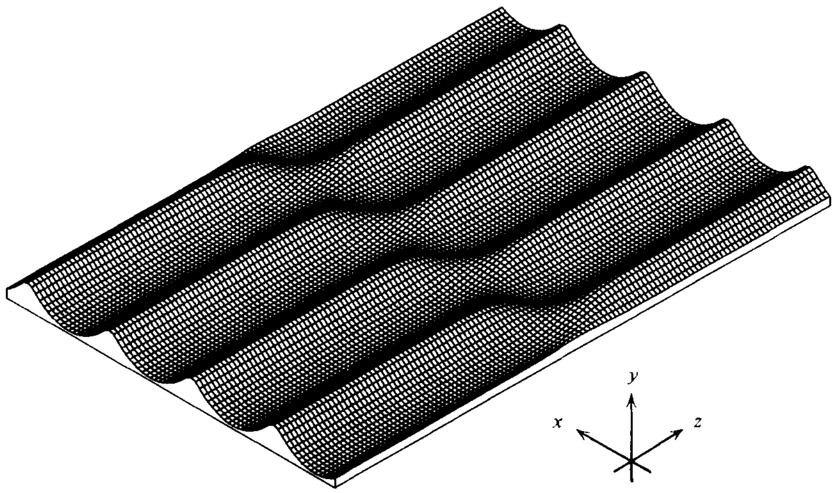

FIGURE 4. Perspective drawing of four wavelengths of the free surface of the semi-infinite-crested wave calculated from the sixth-order solution (4.2) evaluated at a wave steepness of $h = 0.4$.

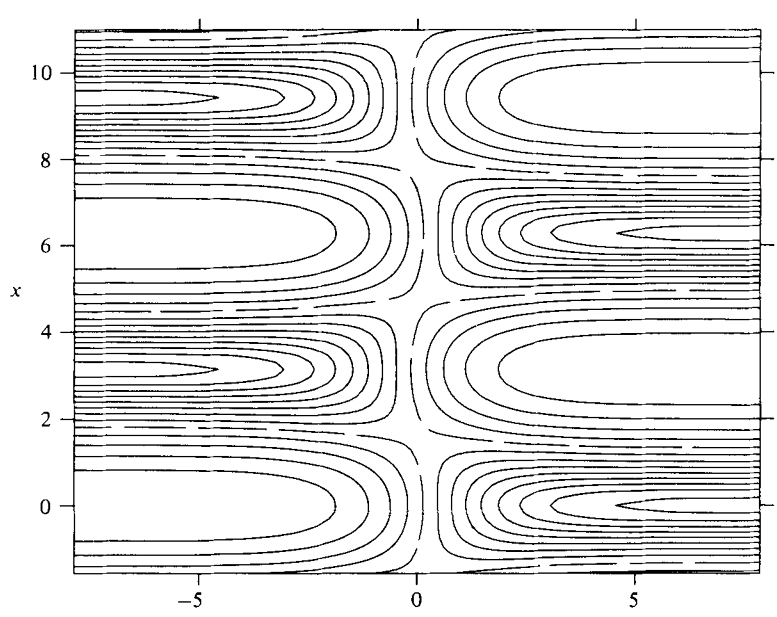

FIGURE 5. Contours of two wavelengths of the same semi-infinite-crested wave as that shown in figure 4; the wave crests continue virtually unchanged to $z = \pm\infty$. The contour interval is 0.06, and the zero-level is shown by the dashed contour.

waves (see their figure 2). However, the nonlinear solutions differ markedly from the linear solution. A uniform wavetrain propagating along the wall with a steadily increasing width is created. Yue & Mei interpret the solution as a jump in wave properties. Peregrine (1983) discusses the structure of the wave jump in more detail. There are several ways of interpreting this structure, but in the present context it is useful to think of the incident waves feeding energy to the waves by the wall and being partially reflected in the process. The resulting wavefield in the growing jump region can then be interpreted at any point as the incident wave plus its partial reflection. Thus the long-crested waves described here are typically not generated by reflection; instead we propose that they will be realized through weak diffraction or refraction effects.

The partial reflection of waves (or, equivalently for linear waves, two wavetrains of arbitrary relative amplitude but the same frequency) forms a more general class of solutions which includes those considered in (I) or above as that special subset in which the amplitudes are equal. The case where two wavetrains only differ slightly in their propagation direction could be incorporated in the present work by adding a term corresponding to a surface elevation $G_1(Z)\sin(X)$ to the linear solution (2.9). However, the more general solutions corresponding to (2.12) are more readily found from wave-modulation equations. For example, the nonlinear Schrödinger equation used in Yue & Mei (1980) has solutions of the form

$$|A|^2 = 1-a_1^2\,\mathrm{cn}^2\,[a_1(Z-cX)/m^{\frac{1}{2}}|m], \tag{5.1}$$

which reduce to solutions of the type (2.12) when $a_1 = 1$, $c = 0$ (for details see the appendix).

Solutions of the form (5.1) and (2.12) indicate that there is a tendency for a wavefield of waves with almost equal wavenumbers to be organized in such a way that a large part of the surface locally has the form of infinite-crested waves. (As $m \rightarrow 1$, the variation of sn and cn is concentrated in a small part of their period.) An important consequence is to confirm the relevance of the extensive analyses that have been done on infinite-crested waves.

The phase variation of solutions (5.1) may be found from (A 4) in the Appendix and is given explicitly in (A 10) for solitons. For long-crested waves, $m \rightarrow 1$, there is a phase change across each minimum of $|A|$. The phase change is equal to one half-period when $|A|$ has a zero; in other cases 'staggered' wave crests occur. These are well illustrated in figure 4 of Su (1982), which gives two photographs of wave patterns arising from the generation of a uniform wavetrain.

The same type of analysis holds for finite water depth; e.g. Yue & Mei (1980) include finite-depth waves in their analysis, and these solutions have the same character. Qualitative features of long-crested waves should be readily observable, particularly in waves approaching a beach where two factors contribute. In deep water the spread of wavenumbers in the wavefield may be broad, but as the waves propagate shorewards they refract and the wavenumbers become much more concentrated in wavenumber space; a situation which may be modelled by two interacting wavetrains propagating in nearly the same direction, i.e. long-crested waves. Also, as the waves approach the beach they steepen (becoming more nonlinear) and thus enhance the observability of the motion (until eventually they break).

The higher-order analysis of §§3 and 4 indicates that the effects of the lower-order nonlinear terms are accentuated. It has not proved possible to proceed far enough to describe waves of near-limiting steepness. The only comparable higher-order work is that of Dysthe (1979), who obtains a modification to the nonlinear Schrödinger equation.

It is almost certain that long-crested deep water waves are subject to the same instabilities as infinite-crested waves; for experimental evidence see Su (1982). A good discussion and survey is given by Yuen & Lake (1982). However, instabilities are of lesser importance in water of shallower depth, and long-crested waves are commonly observed near beaches.

There are more long-crested waves than those described above. By searching for bifurcations of the Stokes-wave solution, Saffman & Yuen (1980) found two sets of solutions of the Zakharov equation. This equation (Zakharov 1968, with minor corrections by Crawford, Saffman & Yuen 1980) is an integrodifferential equation which is of the same order, in its nonlinear approximation, as the analysis of §2 or the nonlinear Schrödinger equation, but allows a wider range of component waves. Such solutions can be more accurate than solutions of the nonlinear Schrödinger equation since harmonics of the fundamental waves are included. On the other hand, they are not as accurate as the special cases dealt with in the body of this paper.

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Appendix

The nonlinear Schrödinger equation used in Yue & Mei (1980) may be written in dimensionless form as

$$2\mathrm{i}A_X + A_{ZZ} - 2|A|^2 A = 0, \tag{A 1}$$

where the linear approximation to the velocity potential is

$$hA(X,Z)\exp[ky + \mathrm{i}(kx - \omega t)] + \text{complex conjugate}, \tag{A 2}$$

in deep water. The frequency of the wave motion is, to this order of approximation, fixed.

Equation (5.1) describes a wave solution whose bulk properties vary only with $\xi = Z - cX$ (hence the modulations of the wavetrain propagate along lines $Z = cX$). To derive such a solution we substitute the form

$$A(X,Z) = a(\xi)\exp[\mathrm{i}\mu X + \mathrm{i}\psi(\xi)], \tag{A 3}$$

where $a(\xi)$ is the amplitude modulation and $\psi(\xi)$ is the phase modulation, into (A 1). This leads to the equations

$$\psi' = c + B_1/a^2, \tag{A 4}$$

$$a'^2 + (\mu + c^2)\,a^2 - a^4 + B_1^2/a^2 = B_2, \tag{A 5}$$

where B_1 and B_2 are constants of integration and μ is introduced to permit the x-wavelength to change with the wave steepness. Equation (A 5) corresponds to (2.11).

The substitution

$$b = a^2 \tag{A 6}$$

allows (A 5) to be written in the form

$$\tfrac{1}{2}b'^2 = 2(b_1 - b)(b_2 - b)(b_2 - b), \tag{A 7}$$

where $b_1 \geqslant b_2 \geqslant b_3 \geqslant 0$. This has the solution

$$b = b_2 - (b_2 - b_3)\,\mathrm{cn}^2[(b_1 - b_3)^{\frac{1}{2}}\xi\,|\,m], \tag{A 8}$$

where

$$m = \frac{b_2 - b_3}{b_1 - b_3}. \tag{A 9}$$

In the present context b_2 may be chosen to be unity without loss of generality, and hence (A 8) is equivalent to (5.1).

The phase variation is obtained from (A 4). In the general case $\psi(\xi)$ can be found in terms of an elliptic integral of the third kind. For the soliton case, corresponding to $m = 1$, we find

$$\tan\psi = a_1(1-a_1^2)^{\frac{1}{2}}\tanh(a_1\xi). \tag{A 10}$$

It may be seen from the coefficient of a^2 in (A 5) that for any given solution, for example (A 8), there is an arbitrariness as to how the constants c and μ are chosen. However, there is no intrinsic difference in those solutions for which $\mu + c^2$ has the same value. The arbitrariness corresponds to a (small) variation in the choice of the X-direction.

The above solutions may be compared with those for the nonlinear Schrödinger equation which has the opposite sign to (A 1) for the nonlinear terms. These solutions are given by Chu & Mei (1971).

To compare solution (A 8) with superposed linear waves note that, apart from an amplitude factor, two waves of the same frequency may be put in the form

$$\zeta = \sin\alpha\cos(px - \omega t + qz) + \cos\alpha\sin(px - \omega t - qz), \tag{A 11}$$

without loss of generality, by suitable choice of the coordinate system Oxz. This may be rewritten

$$\zeta = \mathrm{Re}\{A(z)\exp[\mathrm{i}(px - \omega t)]\}, \tag{A 12}$$

where

$$A(z) = -\sin(qz - \alpha) - \mathrm{i}\cos(qz + \alpha); \tag{A 13}$$

that is

$$|A(z)|^2 = (1 + \sin 2\alpha)\left\{1 - \frac{2\sin 2\alpha}{1 + \sin 2\alpha}\cos^2(qz - \tfrac{1}{4}\pi)\right\}. \tag{A 14}$$

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